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New lower bounds for the three-dimensional finite bin packing problem

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Abstract

The three-dimensional finite bin packing problem (3BP) consists of determining the minimum number of large identical three-dimensional rectangular boxes, *bins*, that are required for allocating without overlapping a given set of three-dimensional rectangular items. The items are allocated into a bin with their edges always parallel or orthogonal to the bin edges. The problem is strongly NP-hard and finds many practical applications. We propose new lower bounds for the problem where the items have a fixed orientation and then we extend these bounds to the more general problem where for each item the subset of rotations by 90° allowed is specified. The proposed lower bounds have been evaluated on different test problems derived from the literature. Computational results show the effectiveness of the new lower bounds.
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1. Introduction

The *three-dimensional finite bin packing problem* (3BP) consists of determining the minimum number of large identical three-dimensional rectangular boxes, *bins*, that are required for allocating without overlapping a given set of rectangular items, each with a given size. The items are allocated with their edges always parallel or orthogonal to the bin edges and they can have a fixed orientation or can be rotated by 90°. The 3BP finds many practical applications as it is a simplified version of many real world problems, e.g. container and pallet loading.

The 3BP is a generalization of the well-known *one-dimensional Bin Packing Problem* (1BP), where n items of given weight w_i have to be packed into the minimum

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number of bins of capacity W. Therefore, the 3BP is strongly NP-hard as well as the 1BP (see Garey and Johnson [10]).

In practical applications there exist many versions of the 3BP. In the literature the oriented 3BP is the problem where item rotation is not allowed, while the non-oriented 3BP is the problem where every items can perform all the feasible rotations by 90°. According to the classification of Lodi et al. [12] for the two-dimensional Bin Packing Problem (2BP), we denote the oriented and nonoriented 3BP with 3BP|O|F and 3BP|R|F, respectively. In this paper, we first consider the 3BP|O|F and then we extend our results to a more general version of 3BP where for each item the subset of allowed 90° rotations is specified. Following the aforementioned classification scheme we propose to denote this problem with 3BP|M|F. The 3BP|M|F contains as special cases problems 3BP|O|F and 3BP|R|F.

Heuristic methods for the 3BP|O|F are recently proposed by Faroe et al. [7] and Lodi et al. [13].

Only one exact method for the 3BP|O|F is presented in the literature and it is proposed by Martello et al. [15]. A new lower bound that dominates those proposed by Martello et al. [15] is discussed in Fekete and Schepers [9]. At our knowledge, neither exact methods nor lower bounds are proposed in the literature for problems 3BP|R|F and 3BP|M|F.

Extensive survey on cutting and packing problems can be found in Coffman et al. [3], Coffman et al. [2], Dowsland and Dowsland [4], Dyckhoff and Finke [5], Dyckhoff et al. [6], Lodi et al. [14] and Lodi et al. [11].

In this paper, we propose new lower bounds for problems 3BP|O|F of complexity $O(n^5)$ that dominate the ones proposed by Martello et al. [15] and Fekete and Schepers [8] both having complexity $O(n^2)$. The computational analysis on test problems from the literature shows the effectiveness of the new lower bounds once implemented into the same exact algorithm of Martello et al. [15]. The resulting exact method is able to solve problems unsolved by the original exact algorithm of Martello et al. [15] and it requires on average less computing time.

The remaining of this paper is organized as follows. In Section 2, we give the problem definition and the notation used throughout the paper. In Section 3, we summarize the lower bounds for the 3BP|O|F presented in the literature. New lower bounds for the 3BP|O|F are presented in Section 4. In Section 5, we extend these new lower bounds to the more general problem 3BP|M|F. In Section 6, the computational performance of the new lower bounds is given on test problems derived from the literature.

2. Problem description

An unlimited stock of three-dimensional rectangular bins of size (W, H, D) are given and n three-dimensional rectangular items of sizes (w_j, h_j, d_j), $j \in J = \{1, \ldots, n\}$, are required to be placed into the bins. The objective is to allocate without overlapping all items into the minimum number of bins.

We consider both cases where items cannot be rotated, i.e., 3BP|O|F, and where items can be rotated by 90°, i.e., 3BP|M|F, described in Section 5. We assume that

the sizes of bins and of items are positive integers satisfying $w_j \leqslant W$, $h_j \leqslant H$ and $d_j \leqslant D$, for every item $j \in J$.

We denote with $V = W \times H \times D$ the volume of the bin and with $v_j = w_j \times h_j \times d_j$ the volume of item $j \in J$.

2.1. Definitions

Let I be an instance of a minimization problem P, $Z(I)$ be the value of the optimal solution to I and $L(I)$ be the value provided by a lower bound L. The *worst-case performance ratio* of L is defined as the largest real number ρ such that $\rho \leqslant L(I)/Z(I)$ for all instance I of P (see also [15,16]).

Given a minimization problem P and two different lower bounds L_1 and L_2, L_1 *dominates* L_2 if and only if $L_1(I) \geqslant L_2(I)$ for all instance I of P. Henceforth, when L_1 dominates L_2 we also write $L_1 \geqslant L_2$.

2.2. The continuous lower bound L_0

A simple lower bound L_0 for problems 3BP|O|F and 3BP|M|F, called *continuous lower bound*, can be computed in $O(n)$ time as follows:

$$L_0 = \left\lceil \frac{\sum_{j=1}^{n} v_j}{V} \right\rceil . \tag{1}$$

Martello et al. [1] have shown that for problem 3BP|O|F the optimal solution value in the worst case can be up to 8 times the L_0 value, i.e., the worst-case performance ratio of lower bound L_0 is $\frac{1}{8}$.

3. Lower bounds for the 3BP|O|F

In this section, we survey the lower bounds proposed by Martello et al. [15] and Fekete and Schepers [9] for the 3BP|O|F. These bounds will be used in Sections 4 and 6 to evaluate the quality of the new proposed lower bounds.

3.1. The lower bound by Martello et al.

In this section, we briefly describe the lower bounds L_1 and L_2 proposed by Martello et al. [15].

3.1.1. Lower bound L_1

The lower bound L_1 makes use of the lower bound L_{1BP} for the one-dimensional Bin Packing Problem (1BP) defined in the following. 1BP is the problem of packing a set S of items into the minimum number of bins of capacity C, where each item $j \in S$ has a weight c_j. A valid lower bound L_{1BP} to 1BP can be computed according the following theorem.

Theorem 1. *Given any integer p, such that $1 \leqslant p \leqslant \frac{1}{2}C$, let $S_1 = \{j \in S\colon c_j > C - p\}$, $S_2 = \{j \in S\colon \frac{1}{2}C < c_j \leqslant C - p\}$ and $S_3 = \{j \in S\colon p \leqslant c_j \leqslant \frac{1}{2}C\}$. A valid lower bound on the optimal 1BP solution value is*

$$L_{1\mathrm{BP}}(S,C) = \max_{1 \leqslant p \leqslant (1/2)C} \{\max\{L_\alpha(p), L_\beta(p)\}\}, \tag{2}$$

where

$$L_\alpha(p) = |S_1 \cup S_2| + \max\left\{0, \left\lceil \frac{\sum_{j \in S_3} c_j + \sum_{j \in S_2} c_j}{C} - |S_2| \right\rceil\right\}, \tag{3}$$

$$L_\beta(p) = |S_1 \cup S_2| + \max\left\{0, \left\lceil \frac{|S_3| - \sum_{j \in S_2} \left\lfloor \frac{C - c_j}{p} \right\rfloor}{\left\lfloor \frac{C}{p} \right\rfloor} \right\rceil\right\}. \tag{4}$$

If the items are sorted according to decreasing weights, $L_{1\mathrm{BP}}(S,C)$ can be computed in $\mathrm{O}(|S|^2)$ time.

Lower bound L_1 is based on the following theorem.

Theorem 2. *Let $J^{WH} = \{j \in J\colon w_j > \frac{1}{2}W$ and $h_j > \frac{1}{2}H\}$. A valid lower bound L_1^{WH} for problem* 3BP|O|F *is obtained by computing the lower bound $L_{1\mathrm{BP}}(S,C)$ for the one-dimensional bin packing problem 1BP-WH where items of $S = J^{WH}$ of weight $c_j = d_j$, $j \in S$, have to be packed into bins of capacity $C = D$.*

Similar lower bound L_1^{HD} is obtained by replacing in J^{WH} and 1BP-WH the width with the depth, i.e., W with D and w_j with d_j. While, lower bound L_1^{WD} is obtained by replacing in J^{WH} and 1BP-WH the height with the depth, i.e., H with D and h_j with d_j. Thus, the overall lower bound can be computed in $\mathrm{O}(n^2)$ time as

$$L_1 = \max\{L_1^{WH}, L_1^{HD}, L_1^{WD}\}. \tag{5}$$

Martello et al. have shown that no dominance relation exists between L_0 and L_1 and that the worstcase performance of L_1 can be arbitrarily bad.

3.1.2. Lower bound L_2

The lower bound L_2 explicitly takes into account the three dimensions of the items and dominates L_1. However, L_2 makes use of L_1^{WH}, L_1^{HD} and L_1^{WD}.

Theorem 3. *Given any pair of integers (p,q), such that $1 \leqslant p \leqslant \frac{1}{2}W$ and $1 \leqslant q \leqslant \frac{1}{2}H$, let*

$$K_1(p,q) = \{j \in J\colon w_j > W - p \text{ and } h_j > H - q\},$$

$$K_2(p,q) = \{j \in J \setminus K_1(p,q)\colon w_j > \tfrac{1}{2}W \text{ and } h_j > \tfrac{1}{2}H\},$$
$$K_3(p,q) = \{j \in J \setminus (K_1(p,q) \cup K_2(p,q))\colon w_j \geqslant p \text{ and } h_j \geqslant q\}. \tag{6}$$

A valid lower bound on the optimal 3BP|O|F *solution value is*

$$L_2^{WH} = L_1^{WH} + \max_{\substack{1 \leqslant p \leqslant (1/2)W \\ 1 \leqslant q \leqslant (1/2)H}} \left\{ \max \left\{ 0, \left\lceil \frac{\sum_{j \in K_2 \cup K_3} v_j + \sum_{j \in K_1} d_j WH}{V} - L_1^{WH} \right\rceil \right\} \right\}. \tag{7}$$

The lower bound L_2^{WH} *can be computed in* $O(n^2)$ *time.*

It is clear that the above results immediately produce two similar lower bounds L_2^{HD} and L_2^{WD}. Lower bound L_2^{HD} is obtained by replacing in (6) and (7) W with D, w_j with d_j and L_1^{WH} with L_1^{HD}. While, L_2^{WD} is obtained by replacing in (6) and (7) H with D, h_j with d_j and L_1^{WH} with L_1^{WD}. Thus, the overall lower bound can be computed in $O(n^2)$ time as

$$L_2 = \max\{L_2^{WH}, L_2^{HD}, L_2^{WD}\}. \tag{8}$$

Martello et al. have shown that L_2 dominates both L_0 and L_1. Therefore, the overall lower bound is $L_{\mathrm{MPV}} = L_2$.

3.2. The lower bound by Fekete and Schepers

Fekete and Schepers [9] have proposed a new lower bound for the 3BP|O|F, called L_{FS}, and have shown that $L_{\mathrm{FS}} \geqslant L_{\mathrm{MPV}}$.

In computing the lower bound for the 3BP|O|F, Fekete and Schepers normalize the item sizes as $\tilde{w}_j = w_j/W$, $\tilde{h}_j = h_j/H$ and $\tilde{d}_j = d_j/D$, $\forall j \in J$, and set $(W,H,D) = (1,1,1)$. Then lower bound L_{FS} is obtained by computing the continuous lower bound L_0 (see Section 2.2) using the volumes of the items transformed by means of the following three dual feasible functions.

Dual feasible function 1: Let $k \in \mathbb{N}$. Then $u^{(k)}(x) = x$, if $(k+1)x \in \mathbb{Z}$, and $u^{(k)}(x) = \lfloor (k+1)x \rfloor \frac{1}{k}$ otherwise.

Dual feasible function 2: Let $\varepsilon \in [0, \frac{1}{2}]$. Then $U^{(\varepsilon)}(x) = 1$, if $x > 1 - \varepsilon$, $U^{(\varepsilon)}(x) = x$, if $\varepsilon \leqslant x \leqslant 1 - \varepsilon$, and $U^{(\varepsilon)}(x) = 0$, if $x < \varepsilon$.

Dual feasible function 3: Let $\varepsilon \in (0, \frac{1}{2}]$. Then $\phi^{(\varepsilon)}(x) = 1 - \lfloor (1-x)\varepsilon^{-1} \rfloor / \lfloor \varepsilon^{-1} \rfloor$, if $x > \frac{1}{2}$, $\phi^{(\varepsilon)}(x) = 1/\lfloor \varepsilon^{-1} \rfloor$, if $\varepsilon \leqslant x \leqslant \frac{1}{2}$, and $\phi^{(\varepsilon)}(x) = 0$, if $x < \varepsilon$.

The lower bound L_{FS} is given by the following theorem.

Theorem 4. *Let* $\alpha, \beta \in (0, \frac{1}{2}]$. *Let*

$$v_j^{(1)(\alpha)} = u^{(1)}(\tilde{w}_j) \times u^{(1)}(\tilde{h}_j) \times U^{(\alpha)}(\tilde{d}_j),$$

$$
\begin{aligned}
v_j^{(2)(\alpha)} &= u^{(1)}(\tilde{w}_j) \times U^{(\alpha)}(\tilde{h}_j) \times u^{(1)}(\tilde{d}_j),\\
v_j^{(3)(\alpha)} &= U^{(\alpha)}(\tilde{w}_j) \times u^{(1)}(\tilde{h}_j) \times u^{(1)}(\tilde{d}_j),\\
v_j^{(4)(\alpha)} &= u^{(1)}(\tilde{w}_j) \times u^{(1)}(\tilde{h}_j) \times \phi^{(\alpha)}(\tilde{d}_j),\\
v_j^{(5)(\alpha)} &= u^{(1)}(\tilde{w}_j) \times \phi^{(\alpha)}(\tilde{h}_j) \times u^{(1)}(\tilde{d}_j),\\
v_j^{(6)(\alpha)} &= \phi^{(\alpha)}(\tilde{w}_j) \times u^{(1)}(\tilde{h}_j) \times u^{(1)}(\tilde{d}_j),\\
v_j^{(7)(\alpha,\beta)} &= U^{(\alpha)}(\tilde{w}_j) \times U^{(\beta)}(\tilde{h}_j) \times \tilde{d}_j,\\
v_j^{(8)(\alpha,\beta)} &= U^{(\alpha)}(\tilde{w}_j) \times \tilde{h}_j \times U^{(\beta)}(\tilde{d}_j),\\
v_j^{(9)(\alpha,\beta)} &= \tilde{w}_j \times U^{(\alpha)}(\tilde{h}_j) \times U^{(\beta)}(\tilde{d}_j).
\end{aligned}
\tag{9}
$$

A valid lower bound on the optimal 3BP|O|F *solution value is*

$$
L_{\mathrm{FS}} = \max\left\{ \max_{\substack{k\in\{1,\dots,6\}\\ 0<\alpha\leqslant 1/2}} \left\{ \left\lceil \sum_{j\in J} v_j^{(k)(\alpha)} \right\rceil \right\} \max_{\substack{k\in\{7,\dots,9\}\\ 0<\alpha\leqslant 1/2,\\ 0<\beta\leqslant 1/2}} \left\{ \left\lceil \sum_{j\in J} v_j^{(k)(\alpha,\beta)} \right\rceil \right\} \right\}. \tag{10}
$$

Moreover, L_{FS} *dominates the lower bound* L_{MPV}.

4. New lower bounds for the 3BP|O|F

In this section, we introduce the new lower bounds L_1^{new} and L_2^{new} for the 3BP|O|F.

4.1. Lower bound L_1^{new}

The new lower bound L_1^{new} takes into account the three dimensions of the items and it is based on the following observations.

Given a triplet of integers (p,q,r), such that $1 \leqslant p \leqslant \frac{1}{2}W$, $1 \leqslant q \leqslant \frac{1}{2}H$ and $1 \leqslant r \leqslant \frac{1}{2}D$, let

$$
\begin{aligned}
&I_1^W(p) = \{j\in J\colon\ w_j > W-p\}, \quad I_2^W(p) = \{j\in J\colon\ p \leqslant w_j \leqslant W-p\},\\
&I_1^H(q) = \{j\in J\colon\ h_j > H-q\}, \quad I_2^H(q) = \{j\in J\colon\ q \leqslant h_j \leqslant H-q\},\\
&I_1^D(r) = \{j\in J\colon\ d_j > D-r\}, \quad I_2^D(r) = \{j\in J\colon\ r \leqslant d_j \leqslant D-r\}.
\end{aligned}
\tag{11}
$$

Note that for every triplet of integers (p,q,r), such that $1 \leqslant p \leqslant \frac{1}{2}W$, $1 \leqslant q \leqslant \frac{1}{2}H$ and $1 \leqslant r \leqslant \frac{1}{2}D$, every item of $I_1(p,q,r) = I_1^W(p) \cap I_1^H(q) \cap I_1^D(r)$ requires a bin. Therefore, $|I_1(p,q,r)|$ is a valid lower bound on the optimal 3BP|O|F solution. Furthermore, this lower bound can be strengthened by observing that items of $I_1(p,q,r)$ cannot be packed together with items of $I_2(p,q,r) = I(p,q,r) \setminus I_1(p,q,r)$, where $I(p,q,r) = \{j \in J\colon w_j \geqslant p, h_j \geqslant q, d_j \geqslant r\}$. Hence, we can improve the lower bound by adding to $|I_1(p,q,r)|$ a lower bound to the number of bins for packing items $I_2(p,q,r)$.

In the following we describe two different lower bounds, called $L_1'(p,q,r)$ and $L_1''(p,q,r)$, to the minimum number of bins required for packing items $I(p,q,r)$.

$L_1'(p,q,r)$ is based on the following observations. Every item $j \in I_1^W(p)$ cannot be packed side by side with any item of $I(p,q,r)$, therefore the bin volume at its left- and right-hand sides cannot be used. Hence, every item $j \in I_1^W(p)$, once placed in a bin, occupies at least a volume equal to Wh_jd_j. Similar considerations can be done for the items of $I_1^H(q)$ and $I_1^D(r)$. Therefore, for each $j \in I(p,q,r)$, an updated volume $v_j'(p,q,r)$ can be computed as follows:

$$v_j'(p,q,r) = w_j'(p)h_j'(q)d_j'(r), \tag{12}$$

where

$$w_j'(k) = \begin{cases} W, & j \in I_1^W(p), \\ w_j & \text{otherwise}, \end{cases} \qquad h_j'(k) = \begin{cases} H, & j \in I_1^H(p), \\ h_j & \text{otherwise}, \end{cases}$$

$$d_j'(k) = \begin{cases} D, & j \in I_1^D(p), \\ d_j & \text{otherwise}. \end{cases}$$

Hence, $L_1'(p,q,r)$ can be computed as a valid lower bound of the one-dimensional bin packing problem 1BP(p,q,r) where items of $S = I(p,q,r)$ of weight $c_j = v_j'(p,q,r)$ have to be packed into bins of capacity $C = V$.

Lower bound $L_1''(p,q,r)$ is based on the following observations. Given the three subsets $I^{WH}(p,q,r) = I_1^W(p) \cap I_1^H(q) \cap I_2(p,q,r)$, $I^{WD}(p,q,r) = I_1^W(p) \cap I_1^D(r) \cap I_2(p,q,r)$ and $I^{HD}(p,q,r) = I_1^H(q) \cap I_1^D(r) \cap I_2(p,q,r)$, two items contained in two of these sets cannot be packed in the same bin. For example for every pair of items $j_1 \in I^{WH}(p,q,r)$ and $j_2 \in I^{HD}(p,q,r)$ we have $w_{j_1} + w_{j_2} > W$, $h_{j_1} + h_{j_2} > H$ and $d_{j_1} + d_{j_2} > D$. Furthermore, items of $I^{WH}(p,q,r)$ can be only packed one behind the other as items of $I^{WD}(p,q,r)$ can be only packed one over the other and items of $I^{HD}(p,q,r)$ can be only packed side by side. Therefore, $L_1''(p,q,r)$ can be computed as follows: $L_1''(p,q,r) = |I_1(p,q,r)| + \bar{L}_1^{WH}(p,q,r) + \bar{L}_1^{WD}(p,q,r) + \bar{L}_1^{HD}(p,q,r)$, where:

- $\bar{L}_1^{WH}(p,q,r)$ is a valid lower bound to the 1BP defined by setting $C = D$, $S = I^{WH}(p,q,r)$ and $c_j = d_j$, for each item $j \in S$.
- $\bar{L}_1^{WD}(p,q,r)$ is a valid lower bound to the 1BP defined by setting $C = H$, $S = I^{WD}(p,q,r)$ and $c_j = h_j$, for each item $j \in S$.
- $\bar{L}_1^{HD}(p,q,r)$ is a valid lower bound to the 1BP defined by setting $C = W$, $S = I^{HD}(p,q,r)$ and $c_j = w_j$, for each item $j \in S$.

The lower bound L_1^{new} is computed as follows:

$$L_1^{\text{new}} = \max_{\substack{1 \leqslant p \leqslant (1/2)W,\ 1 \leqslant q \leqslant (1/2)H \\ 1 \leqslant r \leqslant (1/2)D}} \{\max\{L_1'(p,q,r), L_1''(p,q,r)\}\}. \tag{13}$$

Note that in expression (13) it is sufficient to consider only the values of p, q and r corresponding to distinct values of $w_j \leqslant W/2$, $h_j \leqslant H/2$ and $d_j \leqslant D/2$, respectively (see also Martello et al. [15]). Moreover, the set of values $\{p\}$, $\{q\}$ and $\{r\}$ to consider in evaluating $L_1'(p,q,r)$ can be further reduced by the following observation. If we have two values p and p' such that $w_j'(p) \leqslant w_j'(p')$, for every item $j \in J$, then value p can be discarded. Similarly, we can reduce the value sets $\{q\}$ and $\{r\}$. It is easy to show that each of the resulting set of values cannot contain more than $n/2$ distinct values. Concerning $L_1''(p,q,r)$, it is sufficient to consider the values $p \in \{w_j \leqslant W/2\colon j \in I_1^H(\frac{1}{2}) \cap I_1^D(\frac{1}{2})\}$, $q \in \{h_j \leqslant H/2\colon j \in I_1^W(\frac{1}{2}) \cap I_1^D(\frac{1}{2})\}$ and $r \in \{d_j \leqslant D/2\colon j \in I_1^W(\frac{1}{2}) \cap I_1^H(\frac{1}{2})\}$.

The lower bound L_1^{new} can be computed in $O(n^3 b)$ time, where b is the complexity of the algorithm used for computing a valid lower bound of the one-dimensional bin packing problem. In case the lower bound $L_{1\text{BP}}(S,C)$ is used, then the overall complexity is $O(n^5)$, while if the continuous lower bound is used then the complexity is $O(n^4)$. The following theorem shows that the lower bound L_1^{new} dominates L_{MPV}.

Theorem 5. *If lower bound L_1^{new} is computed using $L_{1\text{BP}}(S,C)$ then $L_1^{\text{new}} \geqslant L_{\text{MPV}}$.*

Proof. By means of the following properties: (i) $a + \max\{b,c\} = \max\{a+b, a+c\}$, if a, b and c are real; (ii) $a + \lceil b \rceil = \lceil a+b \rceil$, if a is an integer and b is a real; lower bound L_2^{WH} given by expression (7) can be rewritten as

$$L_2^{WH} = \max\{L_1^{WH}, \hat{L}_2^{WH}\}, \tag{14}$$

where

$$\hat{L}_2^{WH} = \max_{\substack{1 \leqslant p \leqslant (\frac{1}{2})W \\ 1 \leqslant q \leqslant (\frac{1}{2})H}} \left\{ \left\lceil \frac{\sum_{j \in K_2(p,q) \cup K_3(p,q)} v_j + \sum_{j \in K_1(p,q)} d_j WH}{V} \right\rceil \right\}. \tag{15}$$

In order to show that $L_1^{\text{new}} \geqslant L_2^{WH}$, we prove that $L_1^{\text{new}} \geqslant L_1^{WH}$ and $L_1^{\text{new}} \geqslant \hat{L}_2^{WH}$.

Consider $L_1'(p,q,r)$ for the case where $p = \frac{1}{2}W$, $q = \frac{1}{2}H$ and $r = 1$. $L_1'(W/2, H/2, 1)$ is computed using $L_{1\text{BP}}(S,C)$ for solving the one-dimensional bin packing problem $1\text{BP}(W/2, H/2, 1)$ where items of $S = I(W/2, H/2, 1)$ of weight $c_j = WHd_j$, if $j \in J^{WH}$, and $c_j = v_j'$, if $j \in S \setminus J^{WH}$, have to be packed into a bins of capacity $C = WHD$. If we ignore the item of $S \setminus J^{WH}$, then the remaining problem is equivalent to the one-dimensional bin packing problem where the items of weight $c_j = d_j$, $j \in J^{WH}$, have to be packed into bins of capacity $C = D$, which corresponds to problem 1BP-WH. Since problem $1\text{BP}(W/2, H/2, 1)$ also considers the items belonging to $S \setminus J^{WH}$, then we have $L_1^{\text{new}} \geqslant L_1'(W/2, H/2, 1) \geqslant L_1^{WH}$.

It is easy to see that $\hat{L}_2^{WH}$ represents the maximum of the continuous lower bounds for the one-dimensional bin packing problems, defined for every pair (p,q) such that $1 \leqslant p \leqslant \frac{1}{2}W$ and $1 \leqslant q \leqslant \frac{1}{2}H$, where items of weight $c_j = WHd_j \leqslant v'_j(p,q,1)$, if $j \in K_1(p,q)$, and $c_j = v_j \leqslant v'_j(p,q,1)$, if $j \in K_2(p,q) \cup K_3(p,q)$, have to be packed into bins of capacity $C = V$. Since, for every pair (p,q), $I(p,q,1) = K_1(p,q) \cup K_2(p,q) \cup K_3(p,q)$ and $c_j \leqslant v'_j(p,q,1)$, for every $j \in I(p,q,1)$, then $L_1^{\text{new}} \geqslant \max\{L'_1(p,q,1)$: $1 \leqslant p \leqslant \frac{1}{2}W,\ 1 \leqslant q \leqslant \frac{1}{2}H\} \geqslant \hat{L}_2^{WH}$.

Therefore, we have that $L_1^{\text{new}} \geqslant L_2^{WH}$ and in a similar way we can show that $L_1^{\text{new}} \geqslant L_2^{WD}$ and $L_1^{\text{new}} \geqslant L_2^{HD}$. Hence, we have $L_1^{\text{new}} \geqslant L_{\text{MPV}} = \max\{L_2^{WH}, L_2^{WD}, L_2^{HD}\}$ which completes the proof. □

Note that in the proof of Theorem 5 it was sufficient to consider only the component $L'_1(p,q,r)$ of lower bound L_1^{new}.

4.2. Lower bound L_2^{new}

The lower bound L_2^{new} is an extension to the 3BP|O|F of the lower bound proposed by Martello and Vigo [16] for the 2BP|O|F. It explicitly takes into account the three dimensions of the items as well as L_1^{new}, described in Sections 4.1. Between L_1^{new} and L_2^{new} no dominance relations hold.

Theorem 6. *Given a triplet of integers (p,q,r), such that $1 \leqslant p \leqslant \frac{1}{2}W$, $1 \leqslant q \leqslant \frac{1}{2}H$ and $1 \leqslant r \leqslant \frac{1}{2}D$, a valid lower bound on the optimal* 3BP|O|F *solution is*

$$L_2^{\text{new}} = \max_{\substack{1 \leqslant p \leqslant (1/2)W, 1 \leqslant q \leqslant (1/2)H \\ 1 \leqslant r \leqslant (1/2)D}} \left\{ \left\lceil \frac{\sum_{j \in J} \mu(j,p,q,r)}{\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor} \right\rceil \right\}, \tag{16}$$

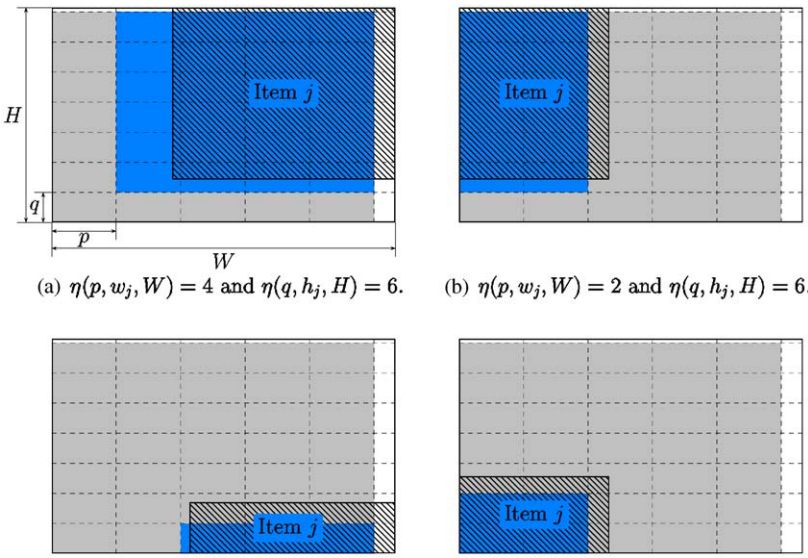
where

$$\mu(j,p,q,r) = \eta(p,w_j,W) \times \eta(q,h_j,H) \times \eta(r,d_j,D) \tag{17}$$

and

$$\eta(s,z,Z) = \begin{cases} \left\lfloor \dfrac{Z}{s} \right\rfloor - \left\lfloor \dfrac{Z-z}{s} \right\rfloor & \text{if } z > \dfrac{Z}{2}, \\ \left\lfloor \dfrac{z}{s} \right\rfloor & \text{if } z \leqslant \dfrac{Z}{2}. \end{cases} \tag{18}$$

Proof. Note that $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$ represents the maximum number of elements of size equal to (p,q,r) that can be packed into a bin. For each item $j \in J$, $\mu(j,p,q,r)$ represents a lower bound on the number of (p,q,r) elements covered by item j (see the four two-dimensional examples reported in Fig. 1). Hence, $\sum_{j \in J} \mu(j,p,q,r)$ represents a lower bound on the number of (p,q,r) elements required for placing all items of J. □



(a) $\eta(p, w_j, W) = 4$ and $\eta(q, h_j, H) = 6$. (b) $\eta(p, w_j, W) = 2$ and $\eta(q, h_j, H) = 6$.

(c) $\eta(p, w_j, W) = 3$ and $\eta(q, h_j, H) = 1$. (d) $\eta(p, w_j, W) = 2$ and $\eta(q, h_j, H) = 2$.

Fig. 1. Four two-dimensional examples: (a) $w_j > W/2$ and $h_j > H/2$; (b) $p \leqslant w_j \leqslant W/2$ and $h_j > H/2$; (c) $w_j > W/2$ and $q \leqslant h_j \leqslant H/2$; (d) $p \leqslant w_j \leqslant W/2$ and $q \leqslant h_j \leqslant H/2$. The lower bound on the number of (p,q) elements covered by item j is given by $\eta(p, w_j, W) \times \eta(q, h_j, H)$.

In order to reduce the computational complexity of L_2^{new} we can reduce the number of triplets (p,q,r) used in expression (16) considering only the values of p, q and r corresponding to distinct values of $w_j \leqslant W/2$, $h_j \leqslant H/2$ and $d_j \leqslant D/2$, respectively. Hence, the lower bound L_2^{new} can be computed in $O(n^4)$ time. Moreover, the set of values $\{p\}$, $\{q\}$ and $\{r\}$ can be further reduced by observing that if there are two values p and p' such that $\eta(p, w_j, W)\lfloor W/p \rfloor^{-1} \leqslant \eta(p', w_j, W)\lfloor W/p' \rfloor^{-1}$, for every item $j \in J$, then value p can be discarded. Similarly, we can reduce the value sets $\{q\}$ and $\{r\}$.

4.3. New lower bound L_B for the 3BP|O|F

Since no dominance relations hold between L_1^{new} and L_2^{new}, but L_1^{new} dominates L_2, the overall lower bound L_B is computed as $L_B = \max\{L_1^{\text{new}}, L_2^{\text{new}}\}$.

Theorem 7. *If lower bounds $L_1'(p,q,r)$ and $L_1''(p,q,r)$ used in expression* (13) *for computing L_1^{new} is computed using the continuous lower bound for the* 1*BP*, *then* $L_B \geqslant L_{FS}$.

Proof. Among the three dual feasible functions proposed by Fekete and Schepers [9] the following relationships hold: $u^{(1)} = U^{(1/2)}$ and $u^{(1)} = \phi^{(1/2)}$. As $U^{(0)}(x) = x$, then

we can rewrite expressions (9) as follows:

$$
\begin{aligned}
v_j^{(1)(\alpha)} &= U^{(1/2)}(\tilde{w}_j) \times U^{(1/2)}(\tilde{h}_j) \times U^{(\alpha)}(\tilde{d}_j),\\
v_j^{(2)(\alpha)} &= U^{(1/2)}(\tilde{w}_j) \times U^{(\alpha)}(\tilde{h}_j) \times U^{(1/2)}(\tilde{d}_j),\\
v_j^{(3)(\alpha)} &= U^{(\alpha)}(\tilde{w}_j) \times U^{(1/2)}(\tilde{h}_j) \times U^{(1/2)}(\tilde{d}_j),\\
v_j^{(4)(\alpha)} &= \phi^{(1/2)}(\tilde{w}_j) \times \phi^{(1/2)}(\tilde{h}_j) \times \phi^{(\alpha)}(\tilde{d}_j),\\
v_j^{(5)(\alpha)} &= \phi^{(1/2)}(\tilde{w}_j) \times \phi^{(\alpha)}(\tilde{h}_j) \times \phi^{(1/2)}(\tilde{d}_j),\\
v_j^{(6)(\alpha)} &= \phi^{(\alpha)}(\tilde{w}_j) \times \phi^{(1/2)}(\tilde{h}_j) \times \phi^{(1/2)}(\tilde{d}_j),\\
v_j^{(7)(\alpha,\beta)} &= U^{(\alpha)}(\tilde{w}_j) \times U^{(\beta)}(\tilde{h}_j) \times U^{(0)}(\tilde{d}_j),\\
v_j^{(8)(\alpha,\beta)} &= U^{(\alpha)}(\tilde{w}_j) \times U^{(0)}(\tilde{h}_j) \times U^{(\beta)}(\tilde{d}_j),\\
v_j^{(9)(\alpha,\beta)} &= U^{(0)}(\tilde{w}_j) \times U^{(\alpha)}(\tilde{h}_j) \times U^{(\beta)}(\tilde{d}_j),
\end{aligned}
\tag{19}
$$

where $\tilde{w}_j = w_j/W$, $\tilde{h}_j = h_j/H$ and $\tilde{d}_j = d_j/D$, $\forall j \in J$.

Part 1: Define $v_j^1(\alpha,\beta,\gamma) = U^{(\alpha)}(\tilde{w}_j) \times U^{(\beta)}(\tilde{h}_j) \times U^{(\gamma)}(\tilde{d}_j)$, where $\alpha,\beta,\gamma \in [0,1/2]$. As $U^{(\varepsilon)}(x)$ is a dual feasible function, then we have the following lower bound L_{P1}:

$$
L_{P1} = \max_{\substack{0 \leqslant \alpha \leqslant 1/2, 0 \leqslant \beta \leqslant 1/2\\ 0 \leqslant \gamma \leqslant 1/2}} \left\{ \left\lceil \sum_{j \in J} v_j^1(\alpha,\beta,\gamma) \right\rceil \right\}. \tag{20}
$$

It is clear that

$$
L_{P1} \geqslant \max \left\{ \max_{\substack{k \in \{1,2,3\}\\ 0 < \alpha \leqslant 1/2}} \left\{ \left\lceil \sum_{j \in J} v_j^{(k)(\alpha)} \right\rceil \right\}, \max_{\substack{k \in \{7,8,9\}\\ 0 < \alpha \leqslant 1/2\\ 0 < \beta \leqslant 1/2}} \left\{ \left\lceil \sum_{j \in J} v_j^{(k)(\alpha,\beta)} \right\rceil \right\} \right\}. \tag{21}
$$

Consider the set $I(p,q,r) = \{j \in J\colon w_j \geqslant p, h_j \geqslant q, d_j \geqslant r\}$ defined in Section 4.1. Let $p = \alpha W$, $q = \beta H$ and $r = \gamma D$ and define $v_j'(p,q,r)$ according to expression (12) for every $j \in I(p,q,r)$ and $v_j'(p,q,r) = 0$ otherwise. From the definition of $U^{(\varepsilon)}(x)$ is easy to see that $v_j^1(\alpha,\beta,\gamma) = v_j'(p,q,r)/V$, for every $j \in I(p,q,r)$, and $v_j^1(\alpha,\beta,\gamma) = 0$ otherwise. Therefore, L_{P1} can be rewritten as follows:

$$
L_{P1} = \max_{\substack{1 \leqslant p \leqslant (1/2)W, 1 \leqslant q \leqslant (1/2)H\\ 1 \leqslant r \leqslant (1/2)D}} \left\{ \left\lceil \sum_{j \in I(p,q,r)} \frac{v_j'(p,q,r)}{V} \right\rceil \right\}. \tag{22}
$$

Note that the term in the maximization of expression (22) is the continuous lower bound of problem 1BP(p,q,r) used in Section 4.1 for computing $L_1'(p,q,r)$. Hence $L_1^{\text{new}} \geqslant L_1'(p,q,r) \geqslant L_{P1}$.

Part 2: Define $v_j^2(\alpha,\beta,\gamma) = \phi^{(\alpha)}(\tilde{w}_j) \times \phi^{(\beta)}(\tilde{h}_j) \times \phi^{(\gamma)}(\tilde{d}_j)$, where $\alpha,\beta,\gamma \in (0,\frac{1}{2}]$. Since $\phi^{(\varepsilon)}(x)$ is a dual feasible function, then we have the following lower bound L_{P2}:

$$L_{P2} = \max_{\substack{0<\alpha\leqslant\frac{1}{2},0<\beta\leqslant\frac{1}{2}\\ 0<\gamma\leqslant\frac{1}{2}}} \left\{ \left\lceil \sum_{j\in J} v_j^2(\alpha,\beta,\gamma) \right\rceil \right\} \geqslant \max_{\substack{k\in\{4,5,6\}\\ 0<\alpha\leqslant\frac{1}{2}}} \left\{ \left\lceil \sum_{j\in J} v_j^{(k)(\alpha)} \right\rceil \right\}. \tag{23}$$

Define $x = z/Z$ and $\varepsilon = s/Z$, then we have $\phi^{(\varepsilon)}(x) = \lfloor Z/s \rfloor - \lfloor (Z-z)/s \rfloor / \lfloor Z/s \rfloor$, if $z > Z/2$, $\phi^{(\varepsilon)}(x) = 1/\lfloor Z/s \rfloor$, if $s \leqslant z \leqslant Z/2$ and $\phi^{(\varepsilon)}(x) = 0$, otherwise. Therefore, L_{P2} can be rewritten as follows:

$$L_{P2} = \max_{\substack{1\leqslant p\leqslant(1/2)W,1\leqslant q\leqslant(1/2)H\\ 1\leqslant r\leqslant(1/2)D}} \left\{ \left\lceil \frac{\sum_{j\in J} \mu'(j,p,q,r)}{\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor} \right\rceil \right\}, \tag{24}$$

where

$$\mu'(j,p,q,r) = \eta'(p,w_j,W) \times \eta'(q,h_j,H) \times \eta'(r,d_j,D) \tag{25}$$

and

$$\eta'(s,z,Z) = \begin{cases} \left\lfloor \dfrac{Z}{s} \right\rfloor - \left\lfloor \dfrac{Z-z}{s} \right\rfloor & \text{if } z > \dfrac{Z}{2}, \\ 1 & \text{if } s \leqslant z \leqslant \dfrac{Z}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

If we consider the lower bound L_2^{new}, defined in Section 4.2, it is easy to show that $\eta'(s,z,Z) \leqslant \eta(s,z,Z)$ and, consequently, $L_2^{\text{new}} \geqslant L_{P2}$.

Hence, we have $L_{\text{B}} = \max\{L_1^{\text{new}}, L_2^{\text{new}}\} \geqslant \max\{L_{P1}, L_{P2}\} \geqslant L_{\text{FS}}$ which completes the proof. □

5. Lower bounds for the 3BP|M|F

For the case where items can be rotated by 90°, we can allocate the item into the bin in six different orientations (see the example reported in Fig. 2). Let $R = \{0,1,2,3,4,5\}$ be the index set of the six rotations. For each rotation $k \in R$ we denote the size of each item $j \in J$ with $(\hat{w}_j(k), \hat{h}_j(k), \hat{d}_j(k))$ where:

$$\hat{w}_j(k) = \begin{cases} w_j, & k \in \{0,1\}, \\ h_j, & k \in \{2,3\}, \\ d_j, & k \in \{4,5\}, \end{cases} \qquad \hat{h}_j(k) = \begin{cases} w_j, & k \in \{2,5\}, \\ h_j, & k \in \{0,4\}, \\ d_j, & k \in \{1,3\}, \end{cases}$$

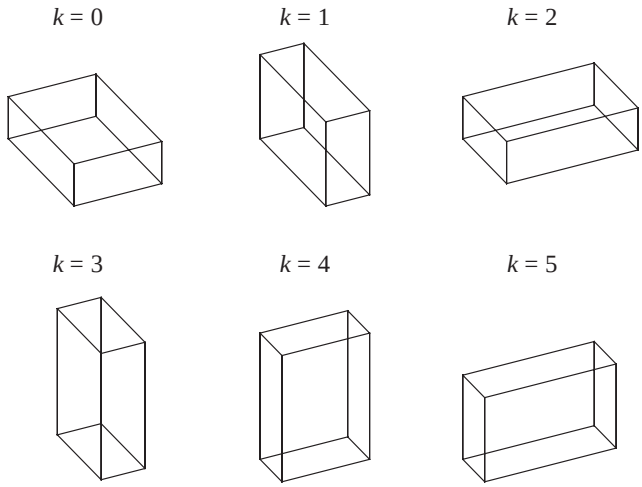


Fig. 2. The six different rotations for item j of size $(w_j, h_j, d_j) = (3, 2, 1)$.

$$\hat{d}_j(k) = \begin{cases} w_j, & k \in \{3,4\}, \\ h_j, & k \in \{1,5\}, \\ d_j, & k \in \{0,2\}. \end{cases}$$

We denote with $R_j \subseteq R$ the subset of feasible rotations allowed of the item $j \in J$; that is $R_j = \{k \in R\colon \hat{w}_j(k) \leqslant W$ and $\hat{h}_j(k) \leqslant H$ and $\hat{d}_j(k) \leqslant D\}$.

Note that in practical application only a subset of R_j can be allowed for item j. In this case, we assume that R_j is specified in input as required by the particular application. If for every item $j \in J$ the subset R_j contains all the feasible rotations, then problem 3BP|M|F corresponds to problem 3BP|R|F.

The continuous lower bound L_0 can be also used for the 3BP|M|F as it involves only item and bin volumes.

5.1. Lower bound L_1^M

The lower bounds described in Section 4 can be used for the 3BP|M|F once the item sizes w_j, h_j and d_j are replaced with the following modified sizes: $\bar{w}_j = \min\{\hat{w}_j(k)\colon k \in R_j\}$, $\bar{h}_j = \min\{\hat{h}_j(k)\colon k \in R_j\}$ and $\bar{d}_j = \min\{\hat{d}_j(k)\colon k \in R_j\}$, while the item volume v_j, $j \in J$, does not need to be modified. However, lower bound $L'(p,q,r)$, described in Section 4.1, can be improved if in defining problem $1BP(p,q,r)$ we set $c_j = \max\{v_j, v'_j(p,q,r)\}$, for every $j \in S = I(p,q,r)$.

We denote with L_1^M the lower bound obtained as described above.

In the following we describe lower bound L_2^M which explicitly takes into account both dimensions and the specified feasible rotations R_j of each item j.

5.2. Lower bound L_2^M

Lower bound L_2^M is defined by the following theorem.

Theorem 8. *A valid lower bound on the optimal solution value of the* 3BP|M|F *is*

$$L_2^M = \max_{\substack{1 \leqslant p \leqslant (1/2)W, 1 \leqslant q \leqslant (1/2)H \\ 1 \leqslant r \leqslant (1/2)D}} \left\{ \left\lceil \frac{\sum_{j \in J} \mu'(j,p,q,r)}{\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor} \right\rceil \right\}, \tag{27}$$

where $\mu'(j,p,q,r) = \min\{\eta(p,\hat{w}_j(k),W) \times \eta(q,\hat{h}_j(k),H) \times \eta(r,\hat{d}_j(k),D)\colon k \in R_j\}$ *and* $\eta(s,z,Z)$ *is defined by expression* (18).

Lower bound L_2^M *can be computed in* $O(n^4)$ *time.*

Proof. Note that $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$ represents the maximum number of elements of size equal to (p,q,r) that can be packed into a bin. For each item $j \in J$, $\mu(j,p,q,r)$ represents a lower bound on the number of (p,q,r) elements covered by item j. Hence, $\sum_{j \in J} \mu(j,p,q,r)$ represents a lower bound on the number of (p,q,r) elements required for placing all items of J. □

5.3. Overall lower bound L_B^M for the 3BP|M|F

Lower bound L_B^M is maximum between L_1^M, computed using the modified item sizes $(\bar{w}_j, \bar{h}_j, \bar{d}_j)$, $j \in J$, and L_2^M given by expression (27), that is $L_B^M = \max\{L_1^M, L_2^M\}$.

6. Computational results

The algorithms presented in this paper have been implemented in C and run on a Pentium III Intel 933 MHz. We have considered the eight classes of test problems proposed by Martello et al. [15].

The first five classes of randomly generated problems that are generalizations of the instances considered by Martello and Vigo [16] for the two-dimensional bin packing problem (2BP). The bin size is $(W,H,D) = (100,100,100)$, and five types of items are considered, as described in Table 1. Each class $k \in \{1,\ldots,5\}$ of instances is obtained by including items of type k with probability 60% and items of the other four types with probability 10% each.

The last three classes of randomly generated problems are generalizations of the instances presented by Berkey and Wang [1] for the 2BP and are defined as follows:

- Class 6: bin size (W,H,D)=(10, 10, 10); items sizes (w_j,h_j,d_j) are uniformly random generated in [1, 10];
- Class 7: bin size (W,H,D)=(40, 40, 40); items sizes (w_j,h_j,d_j) are uniformly random generated in [1, 35];

Table 1
Item generation for Classes 1–5. For each type of item, the dimensions are uniformly random generated in the given range.

Type	w_j	h_j	d_j
1	$[1, \frac{1}{2}W]$	$[\frac{2}{3}H, H]$	$[\frac{2}{3}D, D]$
2	$[\frac{2}{3}W, W]$	$[1, \frac{1}{2}H]$	$[\frac{2}{3}D, D]$
3	$[\frac{2}{3}W, W]$	$[\frac{2}{3}H, H]$	$[1, \frac{1}{2}D]$
4	$[\frac{1}{2}W, W]$	$[\frac{1}{2}H, H]$	$[\frac{1}{2}D, D]$
5	$[1, \frac{1}{2}W]$	$[1, \frac{1}{2}H]$	$[1, \frac{1}{2}D]$

- Class 8: bin size $(W, H, D) = (100, 100, 100)$; items sizes (w_j, h_j, d_j) are uniformly random generated in $[1, 100]$.

The test problem instances have been generated using the code available on website "http://www.diku.dk/∼pisinger/codes.html" and each class contains 10 different problems.

Table 2 shows the results when items cannot be rotated. The computational results obtained by the new lower bound L_B are compared with the ones obtained by the lower bound L_{MPV} proposed by Martello et al. [15] and our implementation of the lower bound L_{FS} proposed by Fekete and Schepers [8]. Moreover, Table 2 shows the results obtained by the exact method proposed by Martello et al. [15] (the code is available on the website "http://www.diku.dk/∼pisinger/codes.html") using their lower bound L_{MPV}, our implementation of L_{FS} and the new lower bound L_B within the time limit of 300 s.

Let UB be the best upper bound known. In Table 2 for each test problem we report:

G_{MPV}, G_{FS}, G_B: average percentage ratio between the lower bound value obtained by L_{MPV}, L_{FS} and L_B, respectively, and the best upper bound known, i.e., $G_{MPV} = \frac{UB - L_{MPV}}{UB} \times 100$, $G_{FS} = \frac{UB - L_{FS}}{UB} \times 100$ and $G_B = \frac{UB - L_B}{UB} \times 100$;

T_{MPV}, T_{FS}, T_B: average computing time in Pentium III Intel 933 MHz CPU s, computed over all the solved instances, required by the original exact method of Martello et al. [15] using lower bounds L_{MPV}, L_{FS} and L_B, respectively;

N_{MPV}, N_{FS}, N_B: average number of tree nodes, computed over all the solved instances, required by the original exact method of Martello et al. [15] using lower bounds L_{MPV}, L_{FS} and L_B, respectively;

O_{MPV}, O_{FS}, O_B: number of instances solved to optimality within the time limit of 300 s by the exact method of Martello et al. [15] using lower bounds L_{MPV}, L_{FS} and L_B, respectively.

Table 2
Lower bounds for the 3BP|O|F.

Problem		MPV 2000				FS '97				New lower bound			
Class	n	G_{MPV}	T_{MPV}	N_{MPV}	O_{MPV}	G_{FS}	T_{FS}	N_{FS}	O_{FS}	G_B	T_B	N_B	O_B
I	20	13.62	0.05	154.9	10	6.52	0.04	110.9	10	3.43	0.03	37.5	10
	40	6.50	21.17	1266415.3	9	4.67	34.69	1264027.7	9	2.22	15.35	139873.9	10
	60	8.95	20.11	205616.0	1	7.59	57.56	205616.0	1	7.01	41.08	205616.0	1
	80	7.97	—	—	0	5.21	—	—	0	4.68	125.48	222481.5	2
	100	8.39	—	—	0	6.63	—	—	0	4.74	1.79	1870.0	2
II	20	11.90	0.03	958.9	10	9.90	0.05	952.2	10	4.03	0.05	531.5	10
	40	9.27	8.08	103450.1	8	5.78	14.17	71922.9	8	3.56	6.23	54171.6	8
	60	8.99	69.32	8689786.0	1	7.10	36.25	4346047.5	2	5.14	0.44	1185.0	2
	80	8.59	—	—	0	5.68	13.89	81.0	1	5.20	7.14	81.0	2
	100	8.74	—	—	0	6.47	—	—	0	5.68	0.60	101.0	1
III	20	17.29	0.02	112.1	10	10.52	0.03	39.3	10	7.10	0.02	31.0	10
	40	10.10	77.12	5440608.1	8	6.11	76.21	4948050.6	7	5.11	53.11	3776889.9	8
	60	8.54	—	—	0	4.56	97.84	104608.0	3	3.72	51.34	78471.3	4
	80	7.85	0.99	81.0	1	6.45	1.10	81.0	1	5.55	1.21	81.0	1
	100	9.19	—	—	0	7.34	—	—	0	6.65	—	—	0
IV	20	0.77	0.01	1.9	10	0.77	0.01	1.9	10	0.00	0.01	0.0	10
	40	2.10	0.05	572.0	10	1.28	0.05	462.6	10	0.00	0.05	0.0	10
	60	2.02	5.24	333185.7	9	0.87	0.20	3619.7	9	0.00	0.13	40.6	10
	80	2.65	1.29	41969.5	4	1.02	0.35	1899.5	6	0.00	0.24	0.0	10
	100	2.39	0.38	97.0	1	0.68	0.40	49.2	6	0.00	0.36	29.5	10
V	20	15.00	0.13	5.2	10	13.33	0.13	5.2	10	0.00	0.12	1.4	10
	40	21.83	0.36	16.0	4	18.58	0.35	16.0	4	10.42	1.58	22.2	5
	60	26.00	2.63	122.0	1	23.64	2.80	122.0	1	16.37	1.97	122.0	1
	80	22.83	—	—	0	22.11	—	—	0	15.56	—	—	0
	100	24.15	—	—	0	22.56	—	—	0	18.23	—	—	0
VI	20	14.10	0.09	12.9	10	12.43	0.09	12.3	10	5.10	0.09	7.5	10
	40	7.33	9.04	3294.9	10	3.50	8.93	3274.1	10	3.50	8.29	3274.1	10
	60	10.23	60.69	131016.0	7	5.60	48.53	127529.7	7	3.98	36.70	111546.9	8
	80	10.27	8.68	168358.8	4	5.43	9.82	168153.3	4	4.31	8.66	163672.0	4
	100	9.35	57.62	3503294.5	2	4.73	59.24	3153301.0	2	4.73	53.06	3153301.0	2
VII	20	14.17	0.11	8.3	10	8.33	0.11	7.2	10	5.00	0.11	6.7	10
	40	30.27	0.22	54.0	2	25.91	0.28	54.0	2	24.48	0.22	54.0	2
	60	28.65	—	—	0	25.63	—	—	0	22.81	—	—	0
	80	26.94	—	—	0	23.82	—	—	0	21.39	—	—	0
	100	26.81	—	—	0	24.34	—	—	0	22.53	—	—	0
VIII	20	13.17	0.09	6.9	10	11.50	0.09	6.9	10	4.17	0.09	4.2	10
	40	12.92	18.10	229.4	8	12.92	18.25	229.4	8	6.13	17.46	208.1	8
	60	13.15	72.60	27518.0	5	11.49	73.76	27518.0	5	9.94	69.56	27518.0	5
	80	18.29	38.61	7953.0	2	16.20	38.78	7953.0	2	12.58	36.44	7953.0	2
	100	14.74	118.89	238339.0	2	13.66	117.69	238339.0	2	9.78	110.68	84240.5	2

We do not report the computing time of the lower bounds as they are negligible.

Table 2 shows that $G_{\mathrm{MPV}} \geqslant G_{\mathrm{FS}} \geqslant G_{\mathrm{B}}$, i.e., $L_{\mathrm{MPV}} \leqslant L_{\mathrm{FS}} \leqslant L_{\mathrm{B}}$, thus confirming the better quality of the new lower bound L_{B}. Not reported in Table 2, lower bound L_{B} for 234 instances out of 400 improves L_{MPV} of Martello et al. [15] and for 118 instances out of 400 improves our implementation of L_{FS} of Fekete and Schepers [9].

Table 2 also shows that the exact method of Martello et al. [15] using the new lower bound L_{B} is able to solve to optimality 31 new instances no solved by the same exact method using L_{MPV}. While, the exact method using L_{FS} is able to solve to optimality only 12 new instances, but it is not able to solve to optimality one instance solved by Martello et al. Moreover, columns T_{MPV}, T_{FS} and T_{B} show that the computational performance of the exact method using the new lower bound L_{B} is on average better than the computational performance of the exact method using L_{MPV} and L_{FS}. Therefore, the larger theoretical computational complexity of L_{B} with respect to L_{MPV} and L_{FS} is repaid by the advantages obtained by having a better lower bound; i.e., a smaller number of tree nodes required to reach the optimal solution (see columns N_{MPV}, N_{FS} and N_{B}). While, even if L_{MPV} and L_{FS} have the same theoretical computational complexity, i.e., $\mathrm{O}(n^2)$, and L_{FS} dominates L_{MPV}, the lower bound L_{FS} is more time consuming than L_{MPV} but it is not able to reduce enough the number of tree nodes.

In our computational experiments we have also tested the new lower bound L_{B} when the continuous lower bound for the 1BP is used, instead of $L_{1\mathrm{BP}}(S, C)$, so as to achieve the $\mathrm{O}(n^4)$ complexity. The results obtained are similar to the ones reported in Table 2 for the version of complexity $\mathrm{O}(n^5)$ but they are always inferior.

7. Conclusions

In this paper, we propose new lower bounds for the three-dimensional finite bin-packing problem where items have a fixed orientation (3BP|O|F) and for the more general case where for each item the subset of rotations allowed is specified (3BP|M|F).

We show that the new lower bound for the 3BP|O|F dominates all other lower bounds presented in the literature so far.

The computational results show the effectiveness of the new lower bound L_{B} for the 3BP|O|F and indicate that the exact algorithm proposed by Martello et al. [15] using L_{B} solves to optimality more instances and requires on average less computing time than the exact algorithm using L_{MPV} and L_{FS}.

In this paper, no results on the worst-case performance ratio of the new lower bound L_{B} are reported. Therefore, further research is required to give an answer to this important question. In the literature, the worstcase performance ratio is defined only for the well-known continuous lower bound L_0 (see Martello et al. [15]).

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